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1984 J. Phys. A: Math. Gen. 17 47

(http://iopscience.iop.org/0305-4470/17/1/008)

J. Phys. A: Math. Gen. 17 (1984) 47–59. Printed in Great Britain

Coupling, subduction and induction transformations for group representations

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Received 3 August 1982, in final form 1 July 1983

Abstract. The basis of a representation space of any group may be chosen according to its decomposition into irreducible representations of the group and a subgroup chain. Coupling, recoupling, subduction and resubduction factors are each shown to arise as a result of Schur's lemmas and a particular group–subgroup structure. When induction and Mackey's subgroup theorem are considered new transformation factors occur, those of induction and reinduction. Key properties of these factors are given, together with their relationship with double coset matrix elements.

1. Introduction

From the works of Schur, Frobenius and Weyl we have a number of theorems connecting the character theory of the symmetric and the unitary groups. This connection, which we call the Schur–Weyl duality, is made quite apparent through the use of Schur functions (Littlewood 1940). But the duality goes further in that many powerful equations can be established which connect various transformation coefficients of the symmetric groups and the unitary groups.

In later papers we shall continue other authors' recent extensions of the Schur–Weyl duality. For this purpose we require some new results in the theory of transformation coefficients for arbitrary compact groups, particularly in the area of induced representations. In this paper we derive these theorems. In so doing we present in § 2 a new perspective on coupling (or isoscalar) factors, recoupling coefficients, and Kramer's $6f$ symbols, showing that they are all specialised transformation factors. Section 3 introduces some double coset bases. Section 4 reviews the process of inducing from subgroup to group and introduces the induction coefficient. Two other types of transformation factors involving induced representations are given in §§ 5 and 6. Their relationships to the induction coefficient are also given.

2. Remarks on transformation coefficients

For our purposes we take a *representation* Γ of a (finite or compact continuous) group G as a unitary vector space called a *representation space* V_Γ of finite dimension $|\Gamma|$ together with a set of unitary linear operators O_g^Γ which map V into itself

$$O_g^\Gamma\colon V_\Gamma \to V_\Gamma \qquad \forall g \in G \tag{2.1}$$

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and which obey the group properties. Because we restrict ourselves to finite-dimensional unitary spaces, §§ 4, 5 and 6 are restricted to induction of finite groups. A *reducible* representation space is one which contains a proper subspace that is invariant under the action of the group operators. An *irreducible* representation space (or *irrep space*) of a group G is a representation space which is not reducible. An irrep space is thus a minimal invariant subspace of a representation space. The action of the group G on any representation space V_Γ leads to the decomposition into irrep spaces. We write

$$V_\Gamma = V_{\Gamma\gamma} \oplus V_{\Gamma\gamma'} \oplus \ldots. \tag{2.2}$$

Furthermore, from Schur's lemma 1, two different irrep spaces $V_{\Gamma\gamma}$ and $V_{\Gamma\gamma'}$ in V_Γ are said to be equivalent if there exists an invertible linear operator $A: V_{\Gamma\gamma} \to V_{\Gamma\gamma'}$, such that $O_g^{\Gamma\gamma'}A = AO_g^{\Gamma\gamma}$. Equivalent irreps may be labelled with the same irrep label γ. An extra label z must then be included to distinguish the different equivalent irreps. We call Γ and z parentage labels. We reserve lower case Greek letters for irrep labels. The decomposition of V_Γ in (2.2) may be rewritten as

$$V_\Gamma = \bigoplus_{z\gamma} V_{\Gamma z\gamma} \tag{2.3}$$

where the sum is over z and γ.

A basis of the irrep space $V_{\Gamma z\gamma}$ is given as the set

$$\{|\Gamma(G)z\gamma(G)i\rangle: i = 1, \ldots, |\gamma|\} \tag{2.4}$$

where $|\gamma|$ is the dimension of $V_{\Gamma z\gamma}$. The group label G is included as a means to distinguish in later sections representation spaces of isomorphic groups. For this section we shall omit the (G), writing $|\Gamma(G)z\gamma(G)i\rangle$ as $|\Gamma z\gamma i\rangle$. The bases of all irrep spaces are chosen to be orthonormal. The action of all group operators on this basis set determines an irreducible matrix representation (*matrix irrep*):

$$O_g^\Gamma|\Gamma z\gamma i\rangle = |\Gamma z\gamma i'\rangle\langle\Gamma z\gamma i'|O_g^\Gamma|\Gamma z\gamma i\rangle \tag{2.5}$$

(the summation convention used throughout this paper is to sum on indices (Greek or Latin) that occur only once in the bra or raised in a matrix, and only once in a ket or lowered in a matrix). We note that for different equivalent irrep spaces $V_{\Gamma z\gamma}$ and $V_{\Gamma z'\gamma}$ the irrep matrices in (2.5) may be different even though their characters are identical. However, it is always possible to choose the bases of all the different equivalent irrep spaces so that the irrep matrices are identical, that is, the irrep matrices are independent of both Γ and z. We write

$$\langle\Gamma z'\gamma' i'|O_g^\Gamma|\Gamma z\gamma i\rangle = \delta^{z'}{}_z\, \delta^{\gamma'}{}_\gamma\, \gamma(g)^{i'}{}_i. \tag{2.6}$$

We call such a basis of V_Γ a *G basis*.

An alternative G basis (we put on 'hats') can be formed by taking linear combinations of the G basis vectors $|\Gamma z\gamma i\rangle$,

$$|\Gamma\hat{z}\hat{\gamma}\hat{i}\rangle = |\Gamma z\gamma i\rangle\langle\Gamma z\gamma i|\Gamma\hat{z}\hat{\gamma}\hat{i}\rangle. \tag{2.7}$$

The irreducibility (Schur's lemma 1) requires that the transformation be diagonal in γ. In general the two G bases give rise to different irrep matrices. Such bases will be called *inequivalent G bases.* However, if the irrep matrices are identical we term the bases *equivalent G bases.* It is important to note that even though the irrep matrices are identical, two equivalent G bases are not necessarily identical. In fact, by Schur's lemma 2 the transformation in (2.7) between equivalent G bases must be diagonal and independent of i, but is otherwise arbitrary:

$$\langle \Gamma z\gamma i|\Gamma\hat{z}\hat{\gamma}\hat{i}\rangle = \langle \Gamma z\gamma|\Gamma\hat{z}\gamma\rangle \delta^{\gamma}{}_{\hat{\gamma}}\,\delta^{i}{}_{\hat{i}}. \tag{2.8}$$

Because all our bases are chosen orthonormal, the elements $\langle \Gamma z\gamma|\Gamma\hat{z}\gamma\rangle$ are elements of a unitary matrix where z and $\hat{z}$ are row and column indexes respectively (see Butler 1981, pp. 15–8).

Now consider the irrep spaces of G with respect to some subgroup H of G. In general the irrep spaces of G are reducible representation spaces of H. If $\eta, \eta' \ldots$ label the irreps of H, we write the decomposition of $V_{\Gamma z\gamma}$ into irreps as

$$V_{\Gamma z\gamma} = \bigoplus_{a\eta} V_{\Gamma z\gamma a\eta} \tag{2.9}$$

where $a = 1, 2, \ldots, |\gamma\colon\eta|$ is the branching multiplicity label which distinguishes the $|\gamma\colon\eta|$ occurrences of the irrep space labelled by η in $V_{\Gamma z\gamma}$. A basis of $V_{\Gamma z\gamma}$ is given by the set of orthonormal vectors

$$\{|\Gamma(G)z\gamma(G)a\eta(H)j\rangle \equiv |\Gamma z\gamma a\eta j\rangle : a = 1, \ldots, |\gamma\colon\eta|, \eta(H), j = 1, \ldots, |\eta|\}. \tag{2.10}$$

An important special case in which we recover a G basis occurs when H is the identity group E with the single irrep $0(E)$. We have $|\gamma\colon 0| = |\gamma|$ and $|\Gamma z\gamma i01\rangle = |\Gamma z\gamma i\rangle$ where i has replaced a. The group operator action on the basis vectors $|\Gamma z\gamma a\eta j\rangle$ is

$$\begin{aligned} O_g^{\Gamma}|\Gamma z\gamma a\eta j\rangle &= |\Gamma z\gamma a'\eta' j'\rangle\langle \Gamma z\gamma a'\eta' j'|O_g^{\Gamma}|\Gamma z\gamma a\eta j\rangle &&\text{for } g\in G,\\ &= |\Gamma z\gamma a\eta j'\rangle\langle \Gamma z\gamma a\eta j'|O_g^{\Gamma}|\Gamma z\gamma a\eta j\rangle &&\text{for } g\in H. \end{aligned} \tag{2.11}$$

In a similar manner to the definition of a G basis, we define a GH basis as a basis which is simultaneously a G basis and an H basis. Note that an H basis is not required to be a G basis. For both GH and H bases one has for $h\in H$

$$\langle \Gamma z'\gamma' a'\eta' j'|O_h^{\Gamma}|\Gamma z\gamma a\eta j\rangle = \delta^{z'}{}_{z}\,\delta^{\gamma'}{}_{\gamma}\,\delta^{a'}{}_{a}\,\delta^{\eta'}{}_{\eta}\,\eta(h)^{j'}{}_{j} \tag{2.12}$$

and for the GH basis with $g\in G$

$$\langle \Gamma z'\gamma' a'\eta' j'|O_g^{\Gamma}|\Gamma z\gamma a\eta j\rangle = \delta^{z'}{}_{z}\,\delta^{\gamma'}{}_{\gamma}\,\gamma(g)^{a'\eta' j'}{}_{a\eta j}. \tag{2.13}$$

Two GH bases are said to be *equivalent GH bases* if they give rise to identical irrep matrices for both G and H (namely $\gamma(g) = \hat{\gamma}(g)$); otherwise they are said to be *inequivalent GH bases.*

In the remainder of this section, we ignore transformations in the parentage label of G bases or GH bases expressed by (2.7). Instead, we look at the properties of the transformation between inequivalent GH bases but equivalent H bases. The transformation may be written

$$|\Gamma z\hat{\gamma}\hat{a}\hat{\eta}\hat{j}\rangle = |\Gamma z\gamma a\eta j\rangle\langle \gamma a\eta j|\hat{\gamma}\hat{a}\hat{\eta}\hat{j}\rangle. \tag{2.14}$$

Irreducibility (Schur's lemma 1 applied to both G and H) foces $\hat{\gamma} = \gamma$ and $\hat{\eta} = \eta$, and the fact that we are transforming between equivalent H bases (Schur's lemma 2 applies to H) forces $\hat{j} = j$ and independence of j. Hence

$$\langle \gamma a\eta j|\hat{\gamma}\hat{a}\hat{\eta}\hat{j}\rangle = \langle \gamma a\eta|\gamma\hat{a}\eta\rangle\,\delta^{\gamma}{}_{\hat{\gamma}}\,\gamma^{\eta}{}_{\hat{\eta}}\,\delta^{j}{}_{\hat{j}}. \tag{2.15}$$

See Bickerstaff (1980). Thus for such a transformation the irrep matrix elements of (2.12) are changed to

$$\gamma(g)^{\hat{b}'\eta' j'}{}_{\hat{b}\eta j} = \langle \gamma\hat{b}'\eta'|\gamma a'\eta'\rangle\gamma(g)^{a'\eta' j'}{}_{a\eta j}\langle \gamma a\nu|\gamma\hat{b}\eta\rangle \qquad \text{for } g\in G, \tag{2.16}$$

but the transformation leaves invariant the elements $\eta(h)^{j'}{}_{j}$. The property (2.15) of this transformation is of great importance in the study of the Racah–Wigner algebra. We call the associated transformation coefficient a *GH transformation factor* to emphasise its independence from the subgroup basis labels and the parentage labels of the group. With each such factored transformation, we associate the diagram of figure 1. The alternative routes correspond to different *GH* basis vectors which have

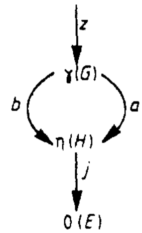

Figure 1.

the same H basis. The transformation factor $\langle \gamma b\eta | \gamma a\eta\rangle$ is the archetype of a class of transformation factors for any two group–subgroup schemes with common group and subgroup. For example, the *GH* transformation factor for the scheme of figure 2 is what we shall call the *coupling factor* and denote as

$$\langle \kappa\mu b\gamma a\eta | (\kappa a_1\lambda ; \mu a_2\nu)c\eta\rangle \equiv \left\langle \begin{array}{c|cc} \kappa\mu & \kappa & \mu \\ b & a_1 & a_2 \\ \gamma & \lambda & \nu \\ a & \multicolumn{2}{c}{c} \\ \eta & \multicolumn{2}{c}{\eta} \end{array} \right\rangle . \tag{2.17}$$

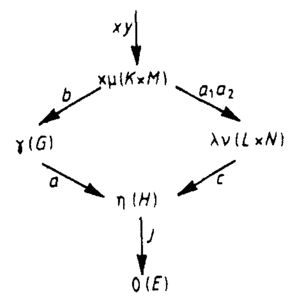

Figure 2.

We observe that if L and N are the identity group E then (2.17) reduces to a *coupling coefficient*

$$\langle \kappa\mu b\gamma i0 | (\kappa k0, \mu m0)10\rangle \equiv \langle \kappa\mu b\gamma i | \kappa k\mu m\rangle \tag{2.18}$$

where a, a_1 and a_2 are replaced by i, k and m respectively. The special branchings $K\times M \supset G$ and $L\times N \supset H$, where K and M are isomorphic to G, and where L and N are isomorphic to H, are termed couplings in the literature on the Racah–Wigner algebra, and hence our use of the terms coupling factor and coupling coefficient.

The scheme of figure 3 (we have omitted the parentage and subgroup basis labels) gives another special transformation factor. This we shall call the *recoupling factor* and denote

$$\langle(\lambda\mu)a\eta,\ \nu,\ b\gamma|\lambda(\mu\nu)c\kappa,\ d\gamma\rangle. \tag{2.19}$$

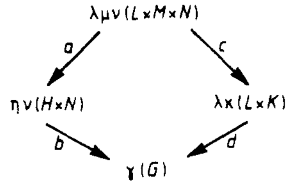

Figure 3.

These factors can be defined with reference to coupling coefficients of (2.18)

$$\langle(\lambda\mu)a\eta,\ \nu,\ b\gamma|\lambda(\mu\nu)c\kappa,\ d\gamma\rangle\langle\lambda\kappa d\gamma i|\lambda l\kappa k\rangle\langle\mu\nu c\kappa k|\mu m\nu n\rangle$$
$$=\langle\eta\nu b\gamma i|\eta j\nu n\rangle\langle\lambda\mu a\eta j|\lambda l\mu m\rangle. \tag{2.20}$$

If all six groups G, H, K, L, M and N are isomorphic, then (2.19) is the recoupling coefficient, which is well known in the Racah–Wigner algebra and which is usually defined by (2.20). See Butler (1981, equation (3.2.17)). To be consistent with our terminology, we call this recoupling coefficient a recoupling factor.

A further example is the factor associated with figure 4 which we write as

$$\langle\gamma a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle. \tag{2.21}$$

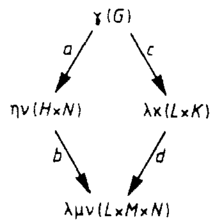

Figure 4.

Kaplan (1962a, b) and Horie (1964) introduced such transformation factors for chains of symmetric groups. Kramer (1967) has analysed and calculated these symmetric group factors, which he termed $6f$ symbols, for all cases without multiplicity. We shall use the term *resubduction factor* for any transformation of the form given by figure 4.

In analogy with the recoupling factor, the resubduction factor can be defined by four transformation coefficients of the type

$$\langle\gamma a\eta j\kappa k|\gamma i\rangle. \tag{2.22}$$

These coefficients describe the decomposition of irreps of G according to the group–subgroup scheme $G\supset H\times K$. Such coefficients will be called *subduction coefficients.* They have been discussed by several authors (see Kramer 1967, 1968), especially in connection with the symmetric groups. The relationship between the resubduction factor and subduction coefficient is given as

$$\langle \gamma a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle\langle\kappa d\mu m\nu n|\kappa k\rangle\langle\gamma c\lambda l\kappa k|\gamma i\rangle$$
$$=\langle\eta b\kappa l\mu m|\eta j\rangle\langle\gamma a\eta j\nu n|\gamma i\rangle \qquad (2.23)$$

and may be used to define the resubduction factor.

To continue the analogy with Racah–Wigner coupling theory, we define the *subduction factor*

$$\langle\gamma c\kappa(a_1\mu)\lambda(a_2\nu)|\gamma a\eta b\mu\nu\rangle \qquad (2.24)$$

which describes the transformation between the group–subgroup schemes given in figure 5. When H is the identity group, (2.24) reduces to a subduction coefficient

$$\langle\gamma c\kappa(k0)\lambda(l0)|\gamma i010\rangle=\langle\gamma c\kappa k\lambda l|\gamma i\rangle \qquad (2.25)$$

where a is replaced by i, a_1 by k, and a_2 by l.

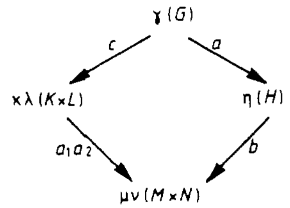

Figure 5.

3. Double coset bases

This section introduces the double coset decomposition which has been studied extensively by Sullivan (1980, and references therein). In contrast to § 2 we study bases of $V_{\Gamma z\gamma}$ for chains involving G and $G_g=gGg^{-1}$ for fixed $g\in G$. Clearly, for any subgroup M of G, $M_g\equiv gMg^{-1}=\{gmg^{-1}:m\in M\}$ is isomorphic to M. We write $M\overset{g}{\sim}M_g$. The case for which g is a double coset representative is of particular interest.

The set $H\backslash G/K$ of *double cosets* HqK of a group G with respect to two subgroups H and K is obtained by writing each element $g\in G$ as

$$g=hqk \qquad (3.1)$$

where $h\in H$ and $k\in K$. The elements q are called *double coset representatives* (Coleman 1966, p 21).

For each q, we have isomorphic subgroups

$$L(q)\equiv H_q\cap K, \qquad L_q(q)\equiv qL(q)q^{-1}, \qquad L_{q^{-1}}(q)\equiv q^{-1}L(q)q. \qquad (3.2)$$

A space V_γ, which is an irrep space of G, is simultaneously an irrep space of G_g for any $g\in G$. But a G basis is not equivalent to a G_g basis because all pairs of matrices $\gamma(g')$ and $\gamma(gg'g^{-1})$ are not equal. Therefore consider transformations between the bases represented by the chains $G\supset K\supset L$, $G\supset H\supset L_{q^{-1}}$, $G_q\supset H_q\supset L$ and $G_q\supset K_q\supset L_q$ as shown in figure 6. (For simplicity we shorten $L(q)$ to L, but note the dependency of L on q.)

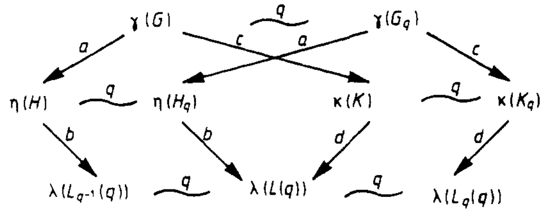

Figure 6.

O_q^γ takes a GKL basis vector (respectively a $GHL_{q^{-1}}$ basis vector) into a $G_qK_qL_q$ basis vector (respectively a G_qH_qL basis vector). That is,

$$O_q^\gamma|\gamma(G)c\kappa(K)d\lambda(L)l\rangle=|\gamma(G_q)c\kappa(K_q)d\lambda(L_q)l\rangle, \tag{3.3}$$

$$O_q^\gamma|\gamma(G)a\eta(H)b\lambda(L_{q^{-1}})l\rangle=|\gamma(G_q)a\eta(H_q)b\lambda(L)l\rangle. \tag{3.4}$$

The overlap between basis vectors of (3.4) and those of the GKL basis defines Sullivan's (1973) *double coset matrix elements* (DCMEs) which clearly have the factorisation property

$$\begin{aligned}\langle\gamma(G)a\eta(H)b\lambda'(L_{q^{-1}})l'|O_q^\gamma|\gamma(G)c\kappa(K)d\lambda(L)l\rangle\\ &=\langle\gamma(G_q)a\eta(H_q)b\lambda'(L)l'|\gamma(G)c\kappa(K)d\lambda(L)l\rangle\\ &=\langle\gamma(G_q)a\eta(H_q)b\lambda(L)|\gamma(G)c\kappa(K)d\lambda(L)\rangle\,\delta^{\lambda'}{}_{\lambda}\,\delta^{l'}{}_{l}.\end{aligned} \tag{3.5}$$

This type of transformation factor has been considered by Reid and Butler (1980, 1982) in their discussion of different (rotated) point group embeddings such as $O\supset D_4\supset C_2$ and $O\supset D_3\supset C_2$ (see also Butler 1981, § 5.3). The resubduction factor of (2.21) can be seen as a special case of a DCME for which in (3.5) $G_q=G$, $H_q=G_{12}\times G_3$, $K=G_1\times G_{23}$, $L=G_1\times G_2\times G_3$ and the bases chosen such that $q=e$. It is not until Mackey's subgroup theorem is introduced in § 6 that we are able to use a powerful completeness relation over the series of subgroups $L(q)$.

4. Bases of induced spaces

The preceding sections discussed transformations arising from the concept of the reduction of an irrep space $V_{z\gamma}$ of a group G into irreps of a subgroup H. A second concept is that of induction, in which a representation space of G is obtained from an irrep space $V_{y\eta}$ of a subgroup H. Induction is the tensor product of $V_{y\eta}$ with the left coset space $V_{H\backslash G}$. Not all physics texts on group theory discuss induced representations, but Coleman (1966, 1968) and Bradley and Cracknell (1972) contain all the results we need.

Recall that the space $V_{H\backslash G}$ is obtained by associating each coset pH with a vector $|p\rangle$ and that the group action in the space $V_{H\backslash G}$ is given by the permutation representation:

$$O_g^{H\backslash G}|p\rangle=|gp\rangle \tag{4.1}$$

The space is of dimension $|H\backslash G|=|G|/|H|$. The induced representation space, $V_{H\backslash G}\otimes V_{y\mu(H)}\equiv V_{y(H)\uparrow G}$ (or written simply as $V_{y\mu\uparrow}$), has the basis vectors

$$|y\eta\uparrow pj\rangle\equiv|y\eta(H_p)j\rangle\equiv|p\rangle|y\eta(H)j\rangle \tag{4.2}$$

where

$$H_p = pHp^{-1}.$$

The basis $\{|p\rangle\}$ of $V_{H\backslash G}$, like the basis $\{|y\eta j\rangle\}$, is not unique, but it is important in the following that once chosen both bases remain fixed.

The action of $g \in G$ on the induced basis vector $|y\eta\uparrow pj\rangle$ is given as

$$O_g^{\eta\uparrow}|y\eta\uparrow pj\rangle = \sum_{p'j'} |y\eta\uparrow p'j'\rangle \eta\uparrow(g)^{p'j'}{}_{pj}. \tag{4.3}$$

Writing gp as the (appropriate) coset representative p_0 times an element $p_0^{-1}gp$ of H, (4.3) becomes

$$\begin{aligned} O_g^{\eta\uparrow}|y\nu\uparrow pj\rangle &= O_{gp}^{\eta\uparrow}|e\rangle|y\eta j\rangle = O_{p_0}^{\eta\uparrow}O_{p_0^{-1}gp}^{\eta\uparrow}|e\rangle|y\eta j\rangle \\ &= |y\eta\uparrow p'j'\rangle\, \delta^{p'}{}_{p_0}\eta(p_0^{-1}gp)^{j'}{}_j. \end{aligned} \tag{4.4}$$

Observe that $\{|y\eta\uparrow pj\rangle\}$ is not a G basis but is a basis of a reducible representation space of G of dimension $|\eta|\times|G|/|H|$. The transformation into a G basis

$$|y\eta\uparrow pj\rangle = |y\eta\uparrow a\gamma i\rangle\langle\eta\uparrow a\gamma i|\eta\uparrow pj\rangle \tag{4.5}$$

gives rise to matrix elements labelled by the index sets pj and $a\gamma i$. We shall call these elements *induction coefficients*. Their factorisation will be given in § 6. The Frobenius reciprocity theorem implies that $a = 1, \ldots, |\gamma\colon\eta|$, that is, the number of occurrences of γ in the induced representation $\eta(H)\uparrow G$ equals the number of occurrences of representation $\eta(H)$ in $\gamma(G)$.

When $\eta(H)$ is the identity irrep $0(E)$ of E, the induced representation $0(E)\uparrow G$ is the $|G|$-dimensional regular representation of G. The reciprocity theorem then implies that if γ is an irrep of G, the multiplicity of γ in $0(E)\uparrow G$ is equal to the multiplicity of $0(E)$ in $\gamma(G)$, which is just the dimension $|\gamma|$ of $\gamma(G)$. Thus it follows that $\Sigma_\gamma |\gamma|^2 = |G|$, and (4.5) is a statement of the decomposition of the regular representation of G into irreps.

5. Reinduction factors

In this section we define the reinduction factor and show its relationship to the induction coefficients. Given the chain $G \supset H \supset L$ the induction of an irrep $\lambda(L)$ into G is equivalent to, that is, gives the same space as, the two-step process of inducing first into H then into G:

$$\lambda(L)\uparrow G \simeq (\lambda(L)\uparrow H)\uparrow G \tag{5.1}$$

(Coleman 1966, theorem 4). Hence, given the chain $G \supset K \supset L$,

$$(\lambda(L)\uparrow H)\uparrow G \simeq (\lambda(L)\uparrow K)\uparrow G, \tag{5.2}$$

the transformation between the bases obtained by (i) induction to the intermediate group (H or K), (ii) decomposition to its irreps (η or κ), (iii) further induction to G and (iv) decomposition into irreps of G gives rise to the transformation factor (see figure 7)

$$\langle\lambda\uparrow b\eta\uparrow a\gamma|\lambda\uparrow d\kappa\uparrow c\gamma\rangle. \tag{5.3}$$

We shall call this factor a *reinduction factor*. In its definition, we have ignored transformations in the parentage label of $\lambda(L)$ and have transformed between

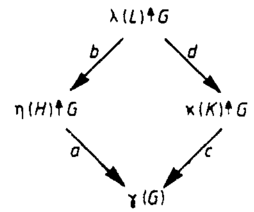

Figure 7.

equivalent G bases. As a notational point observe that the arrows in figure 7 are downward, as in the figures of § 2. This is because the (reducible) representation space $\lambda(L)\uparrow G$ of G may be written as a direct sum of induced representation spaces of either $H\uparrow G$ or $K\uparrow G$, each of which is a direct sum of representation spaces V_γ of G.

The reinduction factor is unitary over the index sets $b\eta a$ and $d\kappa c$. As a result of the Frobenius reciprocity theorem, the transformation factor (see figure 8)

$$\langle\gamma(G)a\eta(H)b\lambda(L)|\gamma(G)c\kappa(K)d\lambda(L)\rangle \tag{5.4}$$

is, for fixed γ and λ, unitary over the same index sets. It is not known if one can make phase and multiplicity choices so that for the same labels the two factors are equal.

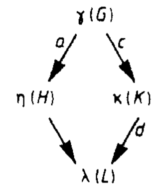

Figure 8.

The reinduction factor can be defined in terms of four induction factors. This is shown by giving alternative bases for the induced space $\lambda(L)\uparrow G$ and using the fact that each element of G can be written $g=p_1h=p_1p_2l'=pl'$ and $g=p_3k=p_3p_4l'=pl'$ where $p_1\in H\backslash G$, $p_2\in L\backslash H$, $p_3\in K\backslash G$, $p_4\in L\backslash K$. Thus

$$\begin{aligned}|\lambda\uparrow pl\rangle &= |\lambda\uparrow p_1p_2l\rangle = |\lambda\uparrow p_1 b\eta j\rangle\langle\lambda\uparrow b\eta j|\lambda\uparrow p_2 l\rangle\\ &= |\lambda\uparrow b\eta\uparrow a\gamma i\rangle\langle\eta\uparrow a\gamma i|\eta\uparrow p_1 j\rangle\langle\lambda\uparrow b\eta j|\lambda\uparrow p_2 l\rangle\end{aligned} \tag{5.5}$$

and similarly

$$\begin{aligned}|\lambda\uparrow pl\rangle &= |\lambda\uparrow p_3p_4l\rangle = |\lambda\uparrow p_3 d\kappa k\rangle\langle\lambda\uparrow d\kappa k|\lambda\uparrow p_4 l\rangle\\ &= |\lambda\uparrow c\kappa\uparrow d\gamma' i'\rangle\langle\kappa\uparrow d\gamma' i'|\kappa\uparrow p_3 k\rangle\langle\lambda\uparrow d\kappa k|\lambda\uparrow p_4 l\rangle.\end{aligned} \tag{5.6}$$

The overlap of these two equations gives the reinduction factor in terms of four induction coefficients:

$$\begin{aligned}&\langle\lambda\uparrow b\eta\uparrow a\gamma|\lambda\uparrow c\kappa\uparrow d\gamma\rangle\langle\kappa\uparrow d\gamma i|\kappa\uparrow p_3 k\rangle\langle\lambda\uparrow d\kappa k|\lambda\uparrow p_4 l\rangle\\ &\quad=\langle\eta\uparrow a\gamma i|\eta\uparrow p_1 j\rangle\langle\lambda\uparrow b\eta j|\lambda\uparrow p_2 l\rangle.\end{aligned} \tag{5.7}$$

The analogy with the recoupling factor which is defined with respect to four coupling coefficients is the reason for our choice of name reinduction factor.

One particular type of reinduction factor which we shall be requiring in later sections is the factor corresponding to figure 9. This will be written as

$$\langle(\lambda\mu)\uparrow a\eta, \nu, \uparrow b\gamma|\lambda(\mu\nu)\uparrow c\kappa, \uparrow d\gamma\rangle \tag{5.8}$$

which by the Frobenius reciprocity theorem has the same unitary properties as the resubduction factor of (2.19) and figure 4. This reinduction factor can also be written as four induction coefficients (cf (5.7))

$$\begin{aligned}\langle(\lambda\mu)\uparrow a\eta, \nu, \uparrow b\gamma|\lambda(\mu\nu)\uparrow c\kappa, \uparrow d\gamma\rangle\langle\lambda\kappa\uparrow d\gamma i|\lambda\kappa\uparrow p_3 lk\rangle\langle\mu\nu\uparrow c\kappa k|\mu\nu\uparrow p_4 mn\rangle \\ =\langle\eta\nu\uparrow b\gamma i|\eta\nu\uparrow p_1 jn\rangle\langle\lambda\mu\uparrow a\eta j|\lambda\mu\uparrow p_2 lm\rangle.\end{aligned}$$

Observe that each induction coefficient G induces from a direct product subgroup, for example in figure 9 $L\times M$ is induced into H.

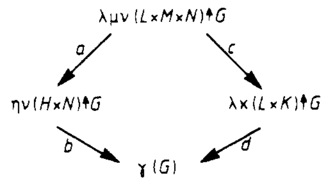

Figure 9.

6. The induction factor and Mackey's subgroup theorem

We can now form two special bases for the induced representation space $V_{\eta(H)\uparrow G}$. The first is the GK basis labelled as

$$\{|\eta\uparrow a\gamma b\kappa k\rangle: a=1,\ldots,|\gamma:\eta|, \gamma(G), b=1,\ldots,|\gamma:\kappa|, \kappa(K), k=1,\ldots,|\kappa|\} \tag{6.1}$$

where, as before, the Frobenius reciprocity theorem gives the range of a.

The second basis is obtained by writing each coset representative p of $H\backslash G$ as $p=rq$ where $q\in H\backslash G/K$ and $r\in L(q)\backslash K$ (see Bradley and Cracknell 1972, theorem 4.7.5), and choosing the $HL_{q^{-1}}(q)$ basis for the space V_η. By writing the H bases in this q-dependent fashion, the basis vectors of $\eta(H)\uparrow G$ may be written

$$\begin{aligned}|\eta\uparrow pj\rangle\equiv|p\rangle|\eta(H)j\rangle &=|rq\rangle|\eta(H)c\lambda(L_{q^{-1}})l\rangle\langle\eta c\lambda l|\eta j\rangle \\ &=|r\rangle|\eta(H_q)c\lambda(L)l\rangle\langle\eta c\lambda l|\eta j\rangle\end{aligned} \tag{6.2}$$

where the q dependence of L must be remembered, and where (3.3) has been used, namely

$$O_q^{\eta\uparrow}|\eta(H)c\lambda(L_{q^{-1}})l\rangle=|\eta(H_q)c\lambda(L)l\rangle. \tag{6.3}$$

The basis vectors

$$\{|r\rangle|\eta(H_q)c\lambda(L)l\rangle: r\in L\backslash K, l=1,\ldots,|\lambda|\} \tag{6.4}$$

for each $qc\lambda$ form an induced space $\lambda(L)\uparrow K$ for which a K basis may be chosen. By this route we have chosen our second basis for the original $\eta(H)\uparrow G$ space. It is

$$
\begin{aligned}
\{|\eta(H_q)c\lambda(L(q))\uparrow d\kappa(K)k\rangle \\
&\equiv |q\eta c\lambda \uparrow d\kappa k\rangle : q \in H\backslash G/K,\ c=1,\ldots,|\eta:\lambda|,\ \lambda(L(q)),\\
&\quad d=1,\ldots,|\lambda:\kappa|,\ \kappa(K),\ k=1,\ldots,|\kappa|\}. \qquad (6.5)
\end{aligned}
$$

The overlap between the basis vectors of (6.1) and (6.5) defines a new transformation factor (see figure 10) which we shall call the *induction factor.*

$$\langle \eta(H_q)c\lambda(L(q))\uparrow d\kappa(K)|\eta(H)\uparrow a\gamma(G)b\kappa(K)\rangle \equiv \langle q\eta c\lambda\uparrow d\kappa|\eta\uparrow a\gamma b\kappa\rangle. \qquad (6.6)$$

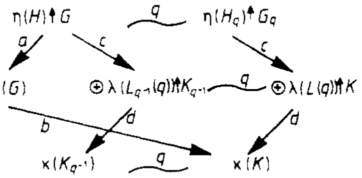

Figure 10.

It is to be emphasised that the vector $|q\eta c\lambda\uparrow d\kappa k\rangle$ belongs to $V_{\eta(H_q)\uparrow G_q}$ through the action of $O_q^{\eta\uparrow}$, which has been absorbed into the vector $|\eta c\lambda\uparrow d\kappa k\rangle$ by (6.3). The factor (6.6) is thus unitary on the index sets $a\gamma b$ and $c\lambda(L(q))d$, where the summation includes all possible subgroups $L(q)$, one for each double coset representative q. The induction factor thus differs from the transformation factors of the earlier sections in that the sum involves groups as well as irreps and multiplicities.

In the language of Mackey's subgroup theorem (Coleman 1966, theorem 8, Bradley and Cracknell 1972, theorem 4.7.6) the induction factor transforms between the representations

$$[\eta(H)\uparrow G]\downarrow K \qquad \text{and} \qquad \bigoplus_{q\in H\backslash G/K} [\eta(H_q)\downarrow L(q)]\uparrow K. \qquad (6.7)$$

We remark that for the case in which K is the identity group, the induction factor is just the induction coefficient defined in (4.4). Here the multiplicity labels b and c label respectively a G basis for $\gamma(G)$ and an H basis for $\eta(H)$. Since $p=eq$, each double coset is a coset of $H\backslash G$. Also $L(q)=E$ for all q since $K\supset L(q)$. Thus (6.6) is rewritten

$$\langle \eta(H_q)c0(E)\uparrow 10(E)|\eta(H)\uparrow a\gamma(G)b0(E)\rangle = \langle \eta\uparrow qc|\eta\uparrow a\gamma b\rangle \qquad (6.8)$$

where we have used (6.2). Comparing this situation with taking $H_1=H_2=H=E$ in figure 2, that is, when the coupling factor reduces to a coupling coefficient, the choice of name for this transformation factor becomes apparent.

In an alternative approach to induction theory using basis projection operators (Young symmetrisers), Sullivan (1973) has shown that the weighted double coset matrix elements (WDCMEs)

$$\left[\frac{|\gamma||\lambda||H||K|}{|G||L||\eta||\kappa|}\right]^{1/2} \langle \gamma(G)a\eta(H)c\lambda(L_{q^{-1}})|O_q^{\gamma}|\gamma(G)b\kappa(K)d\lambda(L)\rangle \qquad (6.9)$$

describe the transformation between the above two basis schemes of the induced representation space. The relationship between the WDCME and our induction factor involves definite phase and multiplicity choices and is not established here.

As an example of the above we take the symmetric groups $H = S_3 \times S_2$, $G = S_5$, $K = S_4 \times S_1$ and the induced space $\mu\uparrow = [21] \times [2]\uparrow$. The coset space $V_{H\backslash G}$, whose representaties p are assumed chosen, has dimension ten so that $|\mu\uparrow| = 20$. The Frobenius reciprocity theorem gives the decomposition of the induced space into G. We have

$$21 \times 2(H)\uparrow G \simeq 41 + 32 + 31^2 + 2^2 1(G). \tag{6.10}$$

where we omit the brackets [ ] around the partition labels. The GK reduction follows simply:

$$41(G) \simeq (4+31) \times 1(K), \qquad 32(G) \simeq (31+2^2) \times 1(K),$$
$$31^2(G) \simeq (31+21^2) \times 1(K), \qquad 2^2 1(G) \simeq (2^2+21^2) \times 1(K). \tag{6.11}$$

The alternative basis is given by first choosing the double cosets of which there are two. Selecting $q_1 = (e)$ and $q_2 = (35)$, the subgroups $L(q)$ of K are respectively

$$L(q_1) = S_3 \times S_1 \times S_0 \times S_1 \qquad \text{and} \qquad L(q_2) = S_2 \times S_2 \times S_1 \times S_0. \tag{6.12}$$

The identity group S_0 is inserted for convenience. The decomposition of $21 \times 2(H_q)$ into each $L(q)$ is

$$21 \times 2(H_{q_1}) \simeq 21 \times 1 \times 0 \times 1(L(q_1))$$

and

$$21 \times 2(H_{q_2}) \simeq 2 \times 2 \times 1 \times 0 + 1^2 \times 2 \times 1 \times 0(L(q_2)). \tag{6.13}$$

The coset spaces $L(q)\backslash K$ are determined by the choices of double cosets q and cosets p, but here we do not need to specify them. We remark that each $V_{L(q)\backslash K}$ is a direct product of two coset spaces since we are performing two inductions, one into S_4 and the other into S_1. The dimensions of the coset spaces are

$$|L(q_1)\backslash K| = 4 \times 1 \qquad \text{and} \qquad |L(q_2)\backslash K| = 6 \times 1. \tag{6.14}$$

Note that $\Sigma_q |L(q)\backslash K| = |H\backslash G|$ which follows from the decomposition $p = rq$.

The resulting decomposition of each induced space $\lambda(L(q))\uparrow K$ is again found by the Frobenius reciprocity theorem. We have

$$\begin{aligned} &21 \times 1 \times 0 \times 1(L(q_1)\uparrow K) \simeq (31 + 2^2 + 21^2) \times 1(K), \\ &2 \times 2 \times 1 \times 0(L(q_2)\uparrow K) \simeq (4 + 31 + 2^2) \times 1(K), \\ &1^2 \times 2 \times 1 \times 0(L(q_2)\uparrow K) \simeq (31 + 21^2) \times 1(K). \end{aligned} \tag{6.15}$$

The direct sum of all the irrep spaces of K is just the composition given in (6.11).

The unitary property of the induction factor is displayed by choosing an irrep of K, say 31×1, and finding the index sets which, in this example, are

$$(a\gamma b) \equiv (\gamma) = (41), (32), (31^2) \tag{6.16}$$

and

$$(c\lambda d) \equiv (\lambda) = (21 \times 1 \times 0 \times 1), (2 \times 2 \times 1 \times 0), (1^2 \times 2 \times 1 \times 0),$$

since the multiplicities a, b, c, d may be omitted.

6. Conclusions

Our desire to study the Schur–Weyl duality relating the symmetric and unitary groups (Haase and Butler 1984) has led us to study transformations between alternative bases of induced representations for arbitrary groups. As a preliminary, we have shown that the well known $3jm$ or coupling and $6j$ or recoupling factors, and the less well known $6f$ or resubduction factor, are all special cases of a general type of basis transformation. All transformations have properties which are a consequence of the conditions imposed under the name, a GH basis choice. The conditions are the irreducibility of spaces, arising from Schur's lemma 1, and the G bases definition which enables us to use Schur's lemma 2.

The possible choices of basis for induced representation spaces has not been studied often. Keeping within the perspectives of the above transformation theory, we have defined the induction coefficient and two new transformation factors, the induction factor and the reinduction factor. We have drawn an analogy with coupling theory in that similar relationships hold for the transformations of induced spaces as for transformations of coupled spaces. The reinduction factor relates two bases of an induced space $V_{\lambda(L)\uparrow G}$, one basis derived from $G \supset H \supset L$ and the other from $G \supset K \supset L$. The induction factor involves both decomposition and induction via Mackey's subgroup theorem. An important feature is the presence of a sum over a set of special subgroups, $L(q) = qHq^{-1} \cap K$ where $q \in H\backslash G/K$ are the double coset representatives.

Acknowledgments

We thank Dr R Paul Bickerstaff and Professor John J Sullivan for many useful discussions.

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